Economics 202
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Lecture # 1

- Mathematics for Eco 202
- Derivative Rules
- Critical Values
- Two-Variable Calculus
  - graphing
  - Derivatives
- Optimization

I. Derivative Rules

1) Power Rule: $f(x) = a x^b$

$$f'(x) = a b x^{b-1}$$

2) Quotient Rule: f and g are diff. functions of X.

$$h(x) = \frac{f(x)}{g(x)} \qquad \frac{dh(x)}{dx} = \frac{f' \cdot g - f \cdot g'}{g^2}$$

3) Product Rule: f and g are diff. functions of X.

$$h(x) = f(x) \cdot g(x) \qquad \frac{dh(x)}{dx} = f' \cdot g + f \cdot g'$$

4) Chain Rule: f and g are diff. functions of x

$$h(x) = g\left(f(x)\right) \qquad \frac{dh(x)}{dx} = g'\left(f(x)\right) \cdot f'(x)$$

5) Special derivatives:

$$f(x) = \ln g(x) \qquad f'(x) = \frac{1}{g(x)} \cdot g'(x)$$

$$f(x) = b^x \qquad f'(x) = \ln b \cdot b^x$$

Special uses of the natural log: → rules apply to log functions generally.

1)
$$f(x_1, x_2) = x_1 \cdot x_2$$
$$\ln f(x_1, x_2) = \ln x_1 + \ln x_2$$

2)
$$f(x_1, x_2) = \frac{x_1}{x_2}$$
$$\ln f(x_1, x_2) = \ln x_1 - \ln x_2$$

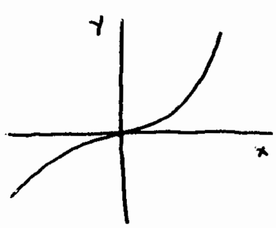
3) $\ln x^5 = 5 \cdot \ln x$

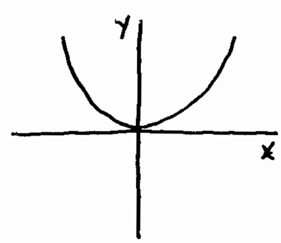
4) $\ln 1 = 0$

II. Critical Values

- First derivatives allow us to find places on a function where the function's slope is zero.

$y = f(x) = x^3$ $\qquad$ $y = f(x) = x^2$





Set the first derivative equal to zero.

$f'(x) = 3x^2$ $\qquad$ $f'(x) = 2x$

$x^2 = 0$ $\qquad$ $2x = 0$

$x^* = 0$ $\qquad$ $x^* = 0$

- Second derivatives help us to check for changes in concavity.

Set the second derivative equal to zero (if possible).

$f''(x) = 6x = 0$ $\qquad$ $f''(x) = 2 > 0$

$x^* = 0$ → concavity changes at $x^* = 0$ $\qquad$ ↳ Always concave up.

III. Two Variable Calculus

In Eco 202 we will see 2-variable functions showing up in consumer theory and producer theory.

Our most common functional forms will be:

Cobb-Douglas : $f(x_1, x_2) = A x_1^{\alpha} x_2^{\beta}$

Perfect Substitutes : $f(x_1, x_2) = a x_1 + b x_2$

Perfect Complements : $f(x_1, x_2) = \text{Min}[a x_1, b x_2]$
(Leontief)

Quasi-Linear : $f(x_1, x_2) = \ln x_1 + x_2$

or

$f(x_1, x_2) = \sqrt{x_1} + x_2$

These are all three dimensional functions. Therefore, we must graph level sets if we are working in two dimensions.

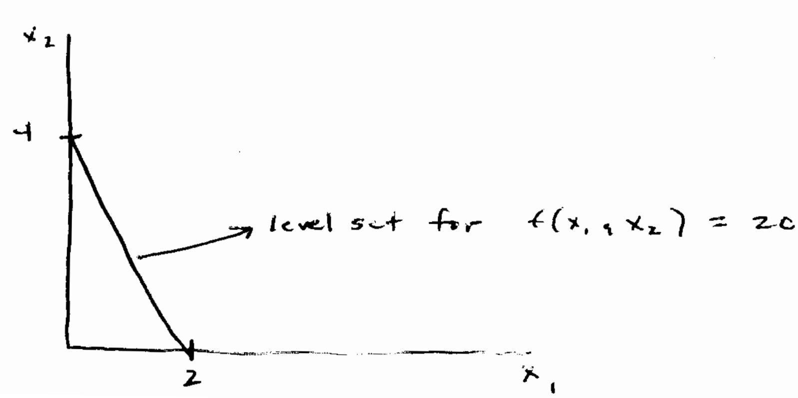
Example : $f(x_1, x_2) = a x_1 + b x_2 = 10 x_1 + 5 x_2$

① Pick a function value. $f(x_1, x_2) = \bar{c}$

② Plot points such that $f(x_1, x_2) = \bar{c}$.

$f(x_1, x_2) = 20$

x_1	x_2
2	0
1	2
$\frac{1}{2}$	3
0	4

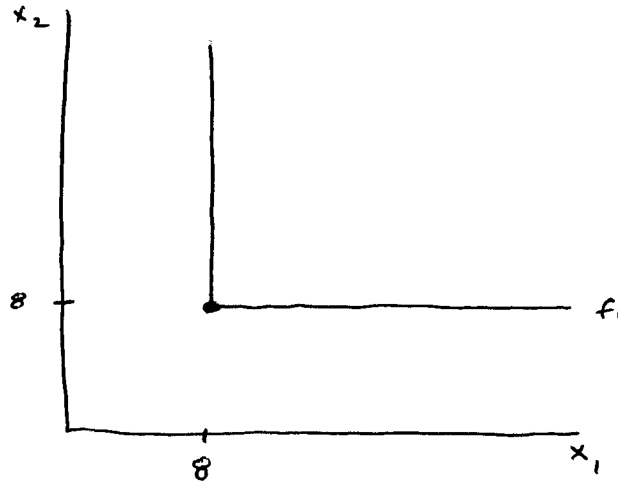


Example 2 :

$f(x_1, x_2) = \text{Min}\,[x_1, x_2]$

$f(x_1, x_2) = 8$

x_1	x_2
8	8
8	10
10	8



Partial Derivatives

$\boxed{\text{Mathematically } \underline{\text{and}} \text{ in Economics!}}$

Partial Derivatives – <u>first</u> what does a "standard" derivative tell us? P. D. tells us the effect of a small change in <u>an</u> independent variable on the dependent variable (holding all other variables constant!).

Example : $f(x_1, x_2) = x_1 x_2$

$\frac{\partial f}{\partial x_1} = x_2 \qquad \frac{\partial f}{\partial x_2} = x_1$

$f(x_1, x_2) = x_1^3 x_2$

$f_1 = 3x_1^2 x_2 \qquad f_2 = x_1^3$

$f(x_1, x_2) = 6x_1 + 10x_2$

$f_1 = 6 \qquad f_2 = 10$

You can apply all of your derivative rules!

IV. Optimization

— Finding the maximum or minimum of a one variable function is fairly straight forward. Doing so for a 2 variable function can be tricky. Fortunately, we want to do constrained optimization.

Idea: How can I maximize my utility given that I face a budget constraint?

Suppose: $U = f(x_1, x_2) = x_1^2 x_2$

$P_1 = \$2$
$P_2 = \$2$ and $I = \$60$

We state the mathematical problem as:

$$\text{Max } U \quad \text{subject to} \quad P_1 x_1 + P_2 x_2 = I$$

$$\Rightarrow \text{Max } x_1^2 x_2 \quad \text{s.t.} \quad 2x_1 + 2x_2 = 60$$

To solve this problem we choose from two metods:

- Substitution
- La Grange's Method

- Substitution

↳ Idea: Turn a two variable function into a one variable function.

$f(x_1, x_2) = x_1^2 x_2$ $\qquad$ $2x_1 + 2x_2 = 60$

$x_2 = 30 - x_1$

$f(x_1) = x_1^2 (30 - x_1)$

$$f'(x_1) = \frac{d[30x_1^2 - x_1^3]}{dx_1} = 60x_1 - 3x_1^2$$

$60x_1 - 3x_1^2 = 0$ $\qquad$ $\left.x_1^* = 0\right\} \rightarrow$ <u>not</u> a maximum. why?

$x_1^* = 20$

↳ $x_2^* = 10$

Is $x_1^* = 20$ $x_2^* = 10$ a maximum?

$f''(x_1) = 60 - 6x_1$

$f''(20) = 60 - 6(20) = -60 < 0$

<u>Yes!</u>

- La Grange's Method

Set up the La Grangian :

$$\mathcal{L} = x_1^2 x_2 - \lambda(2x_1 + 2x_2 - 60)$$

note: budget constraint written in its implicit form.

First Order Conditions :

$$\mathcal{L}_{x_1} = 2x_1 x_2 - 2\lambda = 0$$

$$\mathcal{L}_{x_2} = x_1^2 - 2\lambda = 0$$

$$\mathcal{L}_{\lambda} = -(2x_1 + 2x_2 - 60) = 0$$

$$\left.\begin{array}{l} \frac{2x_1 x_2}{2} = \lambda \\ \frac{x_1^2}{2} = \lambda \end{array}\right\} \rightarrow \frac{2x_1x_2}{2} = \frac{x_1^2}{2} \Rightarrow \frac{2x_2}{x_1} = 1$$

$$\Rightarrow (1) \quad 2x_2 = x_1$$

$$-(2x_1 + 2x_2 - 60) = 0$$

$$\Rightarrow (2) \quad 2x_1 + 2x_2 = 60$$

Plug (1) into (2) $\Rightarrow$ $2x_1 + x_1 = 60$

$$3x_1 = 60$$

$$\boxed{x_1^* = 20 \qquad x_2^* = 10}$$

Where does LaGrange's method run into trouble?

$$f(x_1, x_2) = x_1 + x_2$$

$$f(x_1, x_2) = \text{Min}\{x_1, x_2\}$$

Another use:

Minimization

Ex. Minimize Cost subject to a constraint.

Gadgets are made with widgets and bolts.

$$G = f(W, B) = W \cdot B$$

$$\text{Cost} \Rightarrow \quad P_W \cdot W + P_B \cdot B = \text{Cost}$$

$$P_W = 4 \qquad P_B = 2 \qquad G = 16$$

$$\mathcal{L} = 4 \cdot W + 2B - \lambda(W \cdot B - 16)$$

$$\left.\begin{aligned} \mathcal{L}_W &= 4 - \lambda B = 0 \\ \mathcal{L}_B &= 2 - \lambda W = 0 \end{aligned}\right\} \quad \frac{4}{B} = \frac{2}{W} \rightarrow 4W = 2B$$

$$① \quad 2W = B$$

$$\mathcal{L}_\lambda = -(WB - 16) = 0 \quad \Rightarrow \quad ② \quad W \cdot B = 16$$

$$① \rightarrow ② \qquad W \cdot 2W = 16$$

$$W^2 = 8$$

$$W = \sqrt{8} \qquad B = 2\sqrt{8}$$