

Algorithms for Multivariable Polynomial Interpolation (in Newton Forms)

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Linear Algebra or Numerical Analysis courses often discuss one-variable interpolating polynomials. We show how multivariable algorithms that are not well known are accessible at this level.

For brevity, we focus on the 2-variable quadratic case, though algorithms have been designed and proved for n variables to degree d .

Polynomial interpolation problem:

Does there exist, and how do you find, a unique polynomial

$$p(x,y) = a + bx + cy + dx^2 + exy + fy^2$$

with specified function value at 6 distinct nodes?

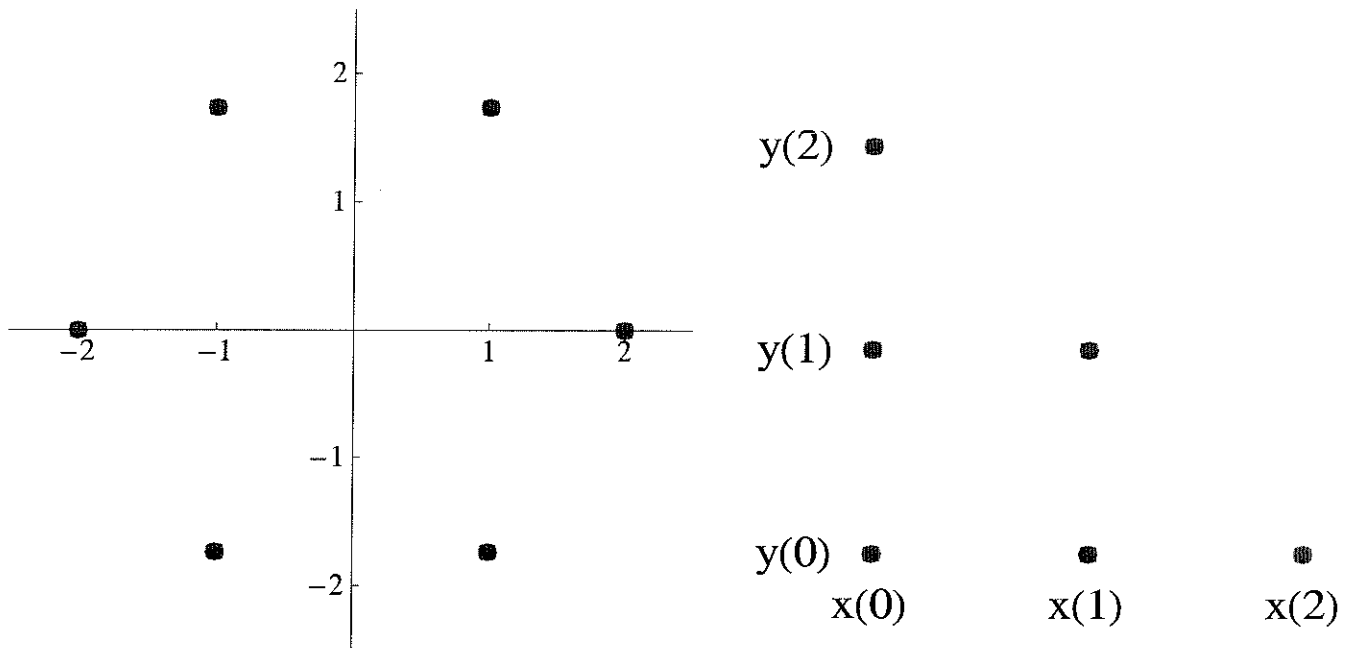
Problem: is a (Vandermonde) matrix M invertible?

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

Students change perspective where x, y have given values and a, b, \dots are unknowns.

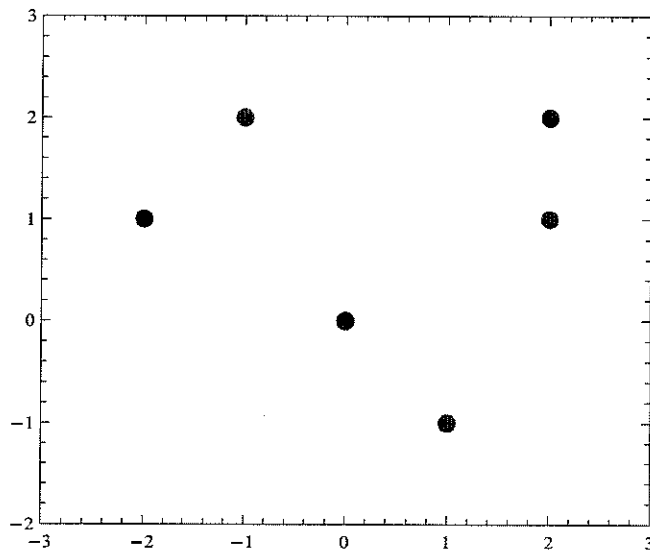
Unlike the single variable, distinct nodes are not enough to give invertibility.

Three examples: circle, corner, generic.



$\exists!$ fails since $x^2 + y^2 = 4$

$\exists!$ poly w/ div-diff



Does $\exists!$ poly?

What can we learn from Gaussian elimination?

Corner is easiest

thanks to a convenient basis:

2	y_2	•		
1	y_1	•	•	
0	y_0	•	•	•
		x_0	x_1	x_2
		0	1	2

classic Newton polynomial basis:

$$q_{00}(x, y) = 1,$$

$$q_{10}(x, y) = x - x_0,$$

$$q_{01}(x, y) = y - y_0,$$

$$q_{20}(x, y) = (x - x_0)(x - x_1)$$

$$q_{11}(x, y) = (x - x_0)(y - y_0),$$

$$q_{02}(x, y) = (y - y_0)(y - y_1).$$

$$\begin{aligned} \text{Span}\{q_{00}, q_{10}, q_{01}, q_{20}, q_{11}, q_{02}\} \\ = \text{Span}\{1, x, y, x^2, xy, y^2\} \end{aligned}$$

Corresponding matrix problem is triangular:

	x_0, y_0	x_1, y_0	x_0, y_1	x_2, y_0	x_1, y_1	x_0, y_2
q_{00}	1	1	1	1	1	1
q_{10}	0	Δx_{10}	0	Δx_{20}	Δx_{10}	0
q_{01}	0	0	Δy_{10}	0	Δy_{10}	Δy_{20}
q_{20}	0	0	0	$\Delta \Delta$	0	0
q_{11}	0	0	0	0	$\Delta \Delta$	0
q_{02}	0	0	0	0	0	$\Delta \Delta$

Simple forward-sub will solve for interpolating polynomial coefficients of Newton basis.

Even simpler – divided difference algorithm works!

	x_0, y_0	x_1, y_0	x_0, y_1	x_2, y_0	x_1, y_1	x_0, y_2
q_{00}	1	1	1	1	1	1
q_{10}	0	Δx_{10}	0	Δx_{20}	Δx_{10}	0
q_{01}	0	0	Δy_{10}	0	Δy_{10}	Δy_{20}
q_{20}	0	0	0	$\Delta \Delta$	0	0
q_{11}	0	0	0	0	$\Delta \Delta$	0
q_{02}	0	0	0	0	0	$\Delta \Delta$

The classic Newton basis polynomials

- (a) are zero at "nodes of lower order"
- (b) are non-zero at the corresponding node and zero at other "nodes of same order"
- (c) have only one term of highest order and completely factor into a product of $(x - a)$ forms

Can we find a generalization of the Newton basis for arbitrary nodes when a unique solution exists?

Algorithm for arbitrary nodes

Form M^T – values of standard basis polynomials at any 6 nodes.

Augment with identity matrix.

	nodes						1	x	y	x^2	xy	y^2
v	1	1	1	1	1	1	1	0	0	0	0	0
a	x_1	x_2	x_3	x_4	x_5	x_6	0	1	0	0	0	0
l	y_1	y_2	y_3	y_4	y_5	y_6	0	0	1	0	0	0
u	*	*	*	*	x_5^2	*	0	0	0	1	0	0
e	*	*	*	*	$x_5 y_5$	*	0	0	0	0	1	0
s	*	*	*	*	y_5^2	*	0	0	0	0	0	1

Elementary row operations preserve the **property:**

Polynomials given by rows of right-half, have values at points in the left-half.

Arbitrary example: -3 third row into fourth row:

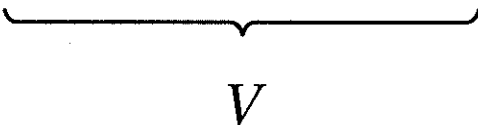
	nodes						1	x	y	x^2	xy	y^2
v	1	1	1	1	1	1	1	0	0	0	0	0
a	x_1	x_2	x_3	x_4	x_5	x_6	0	1	0	0	0	0
l	y_1	y_2	y_3	y_4	y_5	y_6	0	0	1	0	0	0
u	*	*	*	*	$x_5^2 - 3y_5$	*	0	0	-3	1	0	0
e	*	*	*	*	$x_5 y_5$	*	0	0	0	0	1	0
s	*	*	*	*	y_5^2	*	0	0	0	0	0	1

So that fourth row corresponds to $p(x) = -3y + x^2$.

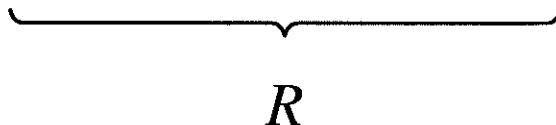
(For now, assume no row exchanges necessary, since that could mess up the desired poly degree.)

The goal: row-reduce to block identity matrices

	nodes						1	x	y	x^2	xy	y^2
v	1	1	1	1	1	1	1	0	0	0	0	0
a	0	1	0	*	*	*	*	*	*	0	0	0
l	0	0	1	*	*	*	*	*	*	0	0	0
u	0	0	0	1	0	0	*	*	*	*	*	*
e	0	0	0	0	1	0	*	*	*	*	*	*
s	0	0	0	0	0	1	*	*	*	*	*	*



V



R

"normalized Newton polynomials" in R

(a) are zero at "nodes of lower order"

(b) are *one* at the corresponding node

and zero at other "nodes of same order"

For a corner of nodes, this algorithm produces the classic Newton polynomials but normalized.

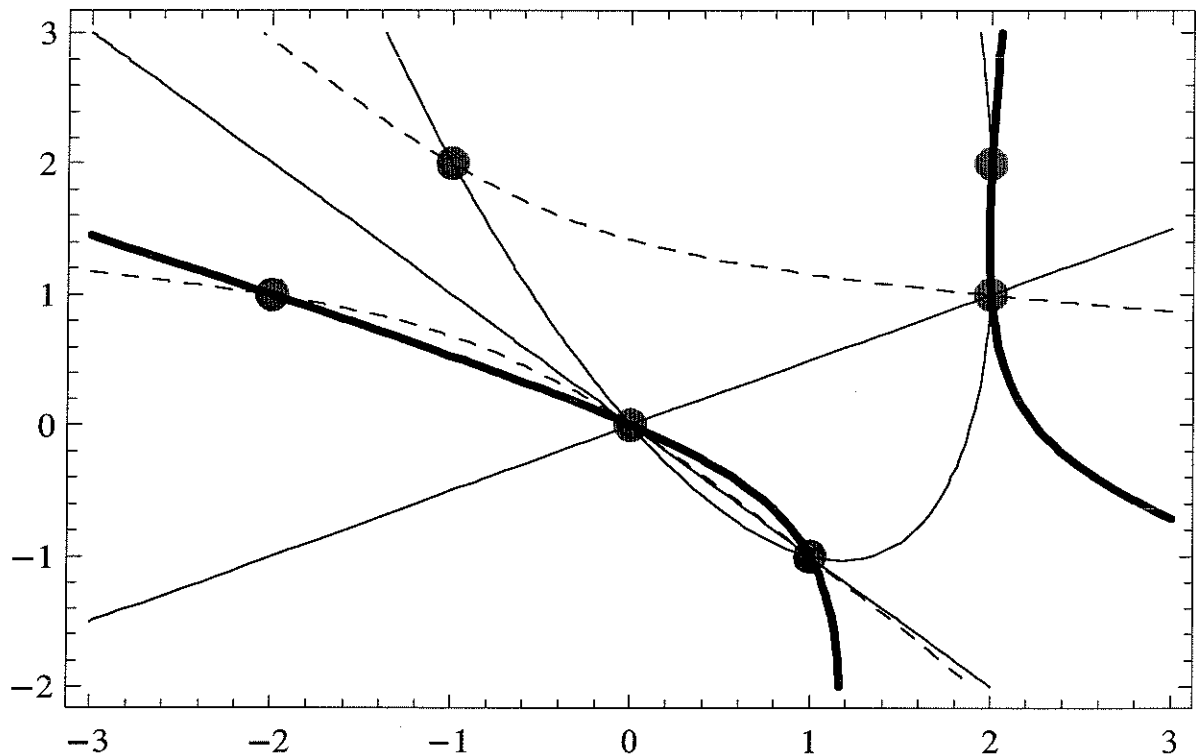
"Generic" Example

nodes						1	x	y	x^2	xy	y^2
1	1	1	1	1	1	1	0	0	0	0	0
0	1	2	2	-1	-2	0	1	0	0	0	0
0	-1	1	2	2	1	0	0	1	0	0	0
0	1	4	4	1	4	0	0	0	1	0	0
0	-1	2	4	-2	-2	0	0	0	0	1	0
0	1	1	4	4	1	0	0	0	0	0	1

reduces to

nodes						1	x	y	x^2	xy	y^2
1	1	1	1	1	1	1	0	0	0	0	0
0	1	0	$\frac{-2}{3}$	$\frac{-5}{3}$	$\frac{-4}{3}$	0	$\frac{1}{3}$	$\frac{-2}{3}$	0	0	0
0	0	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
0	0	0	1	0	0	0	$\frac{-4}{13}$	$\frac{-9}{26}$	$\frac{1}{39}$	$\frac{4}{13}$	$\frac{19}{78}$
0	0	0	0	1	0	0	$\frac{3}{13}$	$\frac{5}{13}$	$\frac{-4}{39}$	$\frac{-3}{13}$	$\frac{1}{39}$
0	0	0	0	0	1	0	$\frac{-17}{52}$	$\frac{-11}{52}$	$\frac{9}{52}$	$\frac{1}{13}$	$\frac{1}{52}$

Example nodes and corresponding polynomials, shown by zero contours, created by the algorithm:

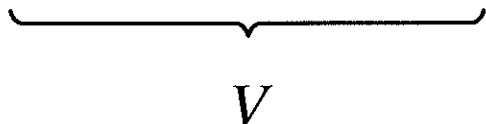


Zeros of normalized Newton polys

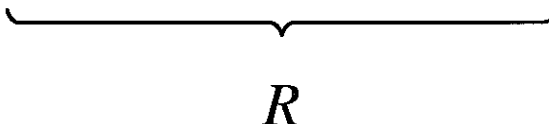
Note: Order of nodes makes a difference if switching between linear and quadratic groups.

Recall:

	nodes						1	x	y	x^2	xy	y^2
v	1	1	1	1	1	1	1	0	0	0	0	0
a	0	1	0	*	*	*	*	*	*	0	0	0
l	0	0	1	*	*	*	*	*	*	0	0	0
u	0	0	0	1	0	0	*	*	*	*	*	*
e	0	0	0	0	1	0	*	*	*	*	*	*
s	0	0	0	0	0	1	*	*	*	*	*	*



V



R

For the interpolating poly for given node values \mathbf{f} , simple forward-sub solves the equation

$$V^T \mathbf{c} = \mathbf{f}$$

for coefficients \mathbf{c} of R -polys.

What if zero is in a pivot position?

Circle of nodes example row-reduces to:

nodes						1	x	y	x^2	xy	y^2
1	1	1	1	1	1	1	0	0	0	0	0
0	1	0	1	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{\sqrt{3}}{6}$	0	0	0
0	0	0	0	3	3	-1	0	0	1	0	0
0	0	0	$4\sqrt{3}$	$\sqrt{3}$	$3\frac{3}{2}$	$\sqrt{3}$	$\sqrt{3}$	1	0	1	0
0	0	0	0	-3	-3	-3	0	0	0	0	1

Exchange columns (not rows) when necessary, to find an ordering of the nodes that is "poised in block" – always possible if M is invertible.

After column exchange, circle example becomes:

nodes						1	x	y	x^2	xy	y^2
1	1	1	1	1	1	1	0	0	0	0	0
0	1	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
0	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{\sqrt{3}}{6}$	0	0	0
0	0	0	1	0	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	0
0	0	0	0	1	1	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{12}$	$\frac{1}{12}$	$\frac{\sqrt{3}}{12}$	0
0	0	0	0	0	0	-4	0	0	1	0	1

Last row gives polynomial $p(x) = -4 + x^2 + y^2$

where all nodes have value zero; finding the circle that makes the unique interpolation impossible!

Theorem (Sauer)

For general set of $\binom{n+d}{d}$ nodes,

row-reduce w/ column exchanges will either:

(a) find an ordering of the nodes with normalized Newton polynomials, or

(b) find an algebraic hypersurface of degree $\leq d$ that contains all the nodes.

The circle of nodes does have $\exists!$ interpolations within a minimal degree polynomial space!

Delete row of zeros and augment with next basis element. For circle example, this x^3 reduces to:

nodes						1	x	y	x^2	xy	x^3
1	1	1	1	1	1	1	0	0	0	0	0
0	1	0	$\frac{-1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
0	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{\sqrt{3}}{6}$	0	0	0
0	0	0	1	0	1	$\frac{-1}{3}$	0	0	$\frac{1}{3}$	0	0
0	0	0	0	1	1	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{12}$	$\frac{1}{12}$	$\frac{\sqrt{3}}{12}$	0
0	0	0	0	0	1	$\frac{-1}{6}$	$\frac{-1}{12}$	0	$\frac{1}{6}$	0	$\frac{1}{12}$

So, for any function values on the circle of nodes, there exists a unique interpolating polynomial in

$$\text{Span}\{1, x, y, x^2, xy, x^3\} =$$

$$\text{Span}\left\{1, \frac{1}{2}(x+1), \frac{1}{6}(\sqrt{3}y+3), \frac{1}{3}(x+1)(x-1),$$

$$\frac{1}{12}(x+1)(x+\sqrt{3}y+2), \frac{1}{12}(x+1)(x-1)(x+2)\right\}$$

Olver Algorithm for arbitrary nodes

To recover some of the algebraic simplicity of classic Newton polynomials, change the block diagonal goal to

invertible matrices (like non-zero in one-dim)
rather than
identity matrices (like 1 in one-dim).

The method is then Block Gaussian Elimination!

For the "generic" example

	nodes					1	x	y	x^2	xy	y^2
1	1	1	1	1	1	1	0	0	0	0	0
0	1	2	2	-1	-2	0	1	0	0	0	0
0	-1	1	2	2	1	0	0	1	0	0	0
0	1	4	4	1	4	0	0	0	1	0	0
0	-1	2	4	-2	-2	0	0	0	0	1	0
0	1	1	4	4	1	0	0	0	0	0	1

pivot on order-one block by

$$\begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^{-1}$$

times order-one block of rows into block of rows below.

Results in:

nodes						1	x	y	x^2	xy	y^2
1	1	1	1	1	1	1	0	0	0	0	0
0	1	2	2	-1	-2	0	1	0	0	0	0
0	-1	1	2	2	1	0	0	1	0	0	0
0	0	0	$\frac{-2}{3}$	$\frac{4}{3}$	$\frac{20}{3}$	0	$\frac{-5}{3}$	$\frac{-2}{3}$	1	0	0
0	0	0	$\frac{2}{3}$	$\frac{-13}{3}$	$\frac{-8}{3}$	0	$\frac{-1}{3}$	$\frac{-4}{3}$	0	1	0
0	0	0	$\frac{10}{3}$	$\frac{16}{3}$	$\frac{8}{3}$	0	$\frac{-2}{3}$	$\frac{1}{3}$	0	0	1

So Newton-Olver polynomial basis is

$$\left\{ 1, x, y, \right. \\ \left. x^2 + \frac{-1}{3}(5x + 2y), \right. \\ \left. xy + \frac{-1}{3}(x + 4y), \right. \\ \left. y^2 + \frac{-1}{3}(2x - y) \right\}$$

The resulting Newton-Olver basis polynomials

(a) are zero at "nodes of lower order."

(b) have values at "nodes of same order" that form an invertible matrix.

(c) have only one term of highest order.

For a corner of nodes, this algorithm produces the classic Newton polynomials.

Conclusion (for Numerical Analysis):

For a (designed) "corner" of nodes, the classic Newton polynomials and divided-difference algorithms generalize to a very efficient method.

For arbitrary nodes, "row-reduce $M^T|I$ with column exchanges" will either find a normalized Newton basis for the nodes or find the hyper-surface on which the nodes lie. For any nodes, extension will find a minimal degree polynomial space with unique interpolating polys.

Block Gaussian elimination produces simpler polynomial basis with (some) Newton properties.

What Linear Algebra students gain:

- linear system where x, y are not unknowns
- interplay of invertibility, uniqueness, existence
- advantage of an alternative basis in poly space
- row and column operations in different context
- minimal degree polynomial subspace
- block Gaussian elimination