

# Minimal Mahler measures

Michael J. Mossinghoff  
Davidson College

Georges Rhin  
Université de Metz

Qiang Wu\*  
Southwest University of China

November 15, 2007

## Abstract

We determine the minimal Mahler measure of a primitive, irreducible, noncyclotomic polynomial with integer coefficients and fixed degree  $D$ , for each even degree  $D \leq 54$ . We also compute all primitive, irreducible, noncyclotomic polynomials with measure less than 1.3 and degree at most 44.

## 1 Introduction

The *Mahler measure* of a polynomial  $P \in \mathbb{C}[X]$ , where

$$P(X) = c_0 X^D + c_1 X^{D-1} + \cdots + c_D = c_0 \prod_{k=1}^D (X - \alpha_k)$$

and  $c_0 \neq 0$ , is defined by

$$M(P) := |c_0| \prod_{k=1}^D \max(1, |\alpha_k|).$$

For an algebraic number  $\alpha$ , we denote by  $M(\alpha)$  the Mahler measure of its minimal polynomial in  $\mathbb{Z}[X]$ . For  $P \in \mathbb{Z}[X]$ , certainly  $M(P) \geq 1$ , and a well-known theorem of Kronecker implies that  $M(P) = 1$  if and only if  $P$  is a product of cyclotomic polynomials and a power of  $X$ . In 1933, D. H. Lehmer [Leh33] asked if there exists a positive number  $\epsilon$  such that if  $\alpha$  is neither 0 nor a root of unity then  $M(\alpha) \geq 1 + \epsilon$ . This is known as *Lehmer's question*, and it remains an open problem. This question arises in a variety of mathematical contexts, including number theory [Sch00, Wal00, Bor02], ergodic theory [Sch95, EW99],

---

\*Supported by SRF for ROCS, SEM (the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry of the Popular Republic of China).

knot theory [Hir01, SW04], and the study of Coxeter systems [McM02, Hir03]. See [GH01, Mos07, Smy07] for additional references.

Lehmer's question has been resolved in some special cases. We say a polynomial is *reciprocal* if  $P(X) = X^D P(1/X)$ , and that an algebraic number is reciprocal if its minimal polynomial is reciprocal. In 1951, Breusch [Bre51] proved that if  $\alpha$  is a nonreciprocal algebraic number, and  $\alpha \notin \{0, 1\}$ , then  $M(\alpha) > 1.179652\dots$ . In 1971, Smyth [Smy71] determined the best possible lower bound in this case, showing that  $M(\alpha) \geq \theta_0 := M(X^3 - X - 1) = 1.324717\dots$ , the smallest Pisot number. Lower bounds have also been established for  $M(\alpha)$  when  $\alpha$  is totally real [Sch73, HS93] or totally  $p$ -adic [BZ01], when the splitting field of  $\alpha$  over  $\mathbb{Q}$  is abelian [AD00] or dihedral [Gar] (and  $\alpha$  is not a root of unity), and when the coefficients of the minimal polynomial of  $\alpha$  satisfy certain arithmetic conditions [BDM07, DM05].

In the general case, the best known explicit result is due to Voutier, who proved in 1996 [Vou96] that if  $\alpha$  is algebraic of degree  $D > 2$  and is not a root of unity, then

$$M(\alpha) > 1 + \frac{1}{4} \left( \frac{\log \log D}{\log D} \right)^3.$$

This is an explicit version of an inequality of Dobrowolski [Dob79].

The smallest known Mahler measure larger than 1 was found by Lehmer in 1933: The polynomial

$$\ell(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

has just one root outside the unit disk, at the real number  $\lambda_0 := 1.176280\dots$  (in fact,  $\lambda_0$  is the smallest known Salem number), so  $M(\ell) = \lambda_0$ . A number of prior computations have established lower bounds on the degree of an algebraic integer  $\alpha$  with  $1 < M(\alpha) < \lambda_0$ , if such a number exists. Boyd [Boy80, Boy89] computed all irreducible, noncyclotomic integer polynomials  $P$  with degree  $D \leq 20$  having  $M(P) < 1.3$ , and the first author [Mos98] used this same algorithm to extend the computation to  $D \leq 24$ . More recently, Flammang, the second author, and Sac-Épée [FRSÉ06] determined all irreducible, noncyclotomic polynomials  $P$  with  $M(P) < \theta_0$  and  $D \leq 36$ , and polynomials  $P$  with  $M(P) < 1.31$  and  $D = 38$  or 40. These computations employed a new method that uses a large family of explicit auxiliary functions to produce improved bounds on the coefficients of polynomials with small Mahler measure.

At a conference in Bristol, U.K., in April 2006, P. Borwein invited the authors to extend these computations to larger degrees on a cluster housed at the centre for Interdisciplinary Research in the Mathematical and Computational Sciences (IRMACS) at Simon Fraser University in British Columbia. Using otherwise idle cycles on the IRMACS workstations, over a period of four months we searched for polynomials with small Mahler measure, accumulating nearly six years of CPU time all together. With these computations, we establish the following theorem. Recall first that a polynomial  $P(X)$  is *primitive* if it cannot be expressed as a polynomial in  $X^k$ , for some  $k \geq 2$ . It is easy to verify that

$M(P(X^k)) = M(P(X))$  for all integers  $k \geq 1$ , so in the following statement we restrict to primitive polynomials.

**Theorem 1** *If a polynomial  $P \in \mathbb{Z}[X]$  satisfies  $1 < M(P) < \lambda_0$ , then  $\deg P \geq 56$ . Further, if  $P$  is an irreducible, noncyclotomic, primitive polynomial of even degree  $D$ , with  $8 \leq D \leq 54$ , then  $M(P) \geq M_D$ , where  $M_D$  is listed for each  $D$  in Table 1.*

$D$	$M_D$	$\nu$	Half of coefficients
8	1.28063816	1	1 0 0 1-1
10	1.17628082	1	1 1 0-1-1-1
12	1.22778556	2	1 1 1 0-1-1-1
14	1.20002652	1	1 0 0 1-1 0 0-1
16	1.22427891	2	1 1 0-1-1 0 1 1 1
18	1.18836815	1	1 1 1 1 0 0-1-1-1-1
20	1.21282418	2	1 1 0 0 1 1 0-1-1-1-1
22	1.20501985	2	1 0 1 0 0 1-1 1 0 0 1-1
24	1.21885515	2	1 0 0 0 0 1 0-1 0 0 0 0-1
26	1.22377745	3	1 1 1 0 0-1-1-1-1 0 0 1 0 1
28	1.20795003	2	1 1 1 1 0 0 0-1-1-1-1 0 0 0 1
30	1.22561985	2	1 0 1 0 0 0 0 0 0 0 0 0 0 1 0 1
32	1.23608337	4	1 1 1 1 0-1-1-2-1 0 0 1 1 0 0 0-1
34	1.22028744	3	1 0 1 0 0 1-1 1-1 0 0-1 1-1 0 0-1 1
36	1.22649330	2	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0-1-1
38	1.22344738	3	1 1 0-1-1 0 0-1-1 0 1 1 0 0 1 1 0-1-1-1
40	1.23624956	3	1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1
42	1.23029547	4	1 0 1 0 0 0-1 1-1 1 0 0 1-1 1-1 1 0 0 1-1 1
44	1.23667481	4	1 0 0 0 0 0 0 1 0 0 0 0 0 0 0-1 0 0 0 0 0-1
46	1.23074301	3	1 0 0 1 0 1 0 0 1 0 0 0 0 0 0-1 0 0-1 0-1 0 0-1
48	1.23220295	4	1 1 0 0 1 1 0-1-1 0 0-1-1 0 1 1 0 0 1 1 0-1-1-1-1
50	1.24037907	3	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0-1-1
52	1.23434837	4	1 1 0-1-1 0 0-1-1 0 1 1 0 0 1 1 0-1-1 0 0-1-1 0 1 1 1
54	1.23656692	5	1 0 1 0 0 0-1 1-1 1 0 0 1-1 1-1 0 0-1 1-1 1-1 0 0-1 1-1

Table 1: Smallest Mahler measures of primitive, irreducible, noncyclotomic polynomials by degree  $D$ . The column labeled  $\nu$  indicates the number of roots outside the unit disk.

Let  $H(P)$  denote the *height* of  $P$ , defined as its largest coefficient in absolute value, so  $H(P) = \max\{|c_k| : 0 \leq k \leq D\}$ . Also, let  $L(P)$  denote its *length*, defined by  $L(P) = \sum_{k=0}^D |c_k|$ .

The first author maintains a website [Mos07] that contains all known irreducible polynomials  $P \in \mathbb{Z}[X]$  with  $\deg P \leq 180$  and  $M(P) < 1.3$ . This includes the polynomials found with the prior exhaustive searches, plus a large number of

polynomials discovered using several heuristic searches. These heuristic methods include testing polynomials with height 1 and a fixed length [Mos98, Lis00], a numerical descent technique [RSÉ03], methods for generating polynomials whose coefficients are in a particular sense close to those of certain cyclotomic polynomials [MPV98, RSÉ03], and checking polynomials associated with certain small limit points of Mahler measures [BM05]. Of these searches, the method of testing polynomials of a particular degree with height 1 and fixed length  $n$  has been remarkably successful. It is well known (see for instance [Mos03]) that if  $M(P) < 2$  then there exists a polynomial  $Q$  in  $\mathbb{Z}[X]$  with  $H(PQ) = 1$ . It is not known if one can always select  $Q$  with  $M(Q) = 1$ , but in practice this appears to be the case, at least for polynomials  $P$  with fairly small degree. Further, it seems that often one can select  $Q$  with very small degree. (It follows from [BV87] that if  $P$  is irreducible,  $\deg P = D$ , and  $M(P) < 2$ , then such a polynomial  $Q$  exists with  $\deg Q = O(\frac{D \log D}{\log 2 - \log M(P)})$ .)

Table 5 in Section 3 summarizes the search on height 1 polynomials performed in [Mos98]. The recent article [FRSÉ06] verified that this search found all polynomials with measure less than 1.3 and degree at most 40. In addition, no heuristic search performed in the last decade found any additional polynomials with measure less than 1.3 and degree less than 174. It is somewhat surprising, then, that the searches we describe here discover three new polynomials with Mahler measure below 1.3 and degree  $D \leq 52$ . These polynomials are listed in Table 4 of Section 3.

Section 2 describes the search method, and Section 3 summarizes our computations and results.

## 2 The Algorithm

For a fixed even integer  $D = 2d$  and a fixed real number  $M \in (1, \theta_0]$ , we wish to determine the set of primitive, irreducible, noncyclotomic polynomials  $P$  with  $\deg P = D$  and  $M(P) < M$ . We describe the algorithm of [FRSÉ06] for calculating this set of polynomials, and our description here follows that paper. The set is computed in three principal steps. First, we determine bounds on certain symmetric functions of the roots of a polynomial  $P$  satisfying  $\deg P = D$  and  $M(P) < M$ . These inequalities determine bounds on the coefficients of an admissible polynomial  $P$ , and we let  $E_1 = E_1(D, M)$  denote the set of polynomials whose coefficients satisfy these bounds. Second, we test each polynomial  $P \in E_1$  against some additional necessary conditions, and the surviving polynomials form a set  $E_2 = E_2(D, M)$ . This step requires the great majority of the computation time. Third, we subject each polynomial  $P \in E_2$  to further tests using Graeffe root squaring, and finally we compute the measures of the surviving polynomials. This produces the set we desire, which we denote by  $E_3 = E_3(D, M)$ .

## 2.1 Notation

By Smyth's theorem, a nonzero algebraic integer with measure less than  $\theta_0$  must be reciprocal. Let  $P$  denote its minimal polynomial, which must have even degree  $2d$ :

$$P(X) = X^{2d} + c_1 X^{2d-1} + \cdots + c_1 X + 1 = \prod_{i=1}^{2d} (X - \alpha_i).$$

We may suppose that  $|\alpha_i| \geq 1$  and  $\alpha_{d+i} = 1/\alpha_i$  for  $1 \leq i \leq d$ . We define a polynomial  $Q$  associated with the polynomial  $P$  by the formula  $X^d Q(X + 1/X) = P(X)$ . Thus,  $Q$  is a monic polynomial of degree  $d$  with integer coefficients,

$$Q(X) = X^d + b_1 X^{d-1} + \cdots + b_{d-1} X + b_d,$$

and its roots are  $\gamma_i := \alpha_i + 1/\alpha_i$  for  $1 \leq i \leq d$ . For  $k \geq 1$ , let

$$\gamma_{i,k} := \alpha_i^k + 1/\alpha_i^k,$$

and let

$$s_k := \sum_{i=1}^d \gamma_{i,k} = \sum_{i=1}^{2d} \alpha_i^k.$$

If we suppose that  $1 \leq |\alpha_i| \leq M^a$  with  $0 < a \leq 1$ , then  $\gamma_i$  lies inside the ellipse

$$\mathcal{E}_a := \left\{ z = x + iy : \left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 \leq 1 \right\},$$

where  $A = M^a + M^{-a}$  and  $B = M^a - M^{-a}$ . Last, we let  $\mathcal{E}_0$  denote the real interval  $[-2, 2]$ .

## 2.2 Bounds on the $s_k$ .

To compute the bounds on the  $s_k$ , we use a family of auxiliary functions, each of the form

$$f(z) = \operatorname{Re}(z) - \sum_{j=1}^J e_j \log |Q_j(z)| \geq m. \quad (1)$$

Here,  $z$  is a complex number in the ellipse  $\mathcal{E}_a$ , the  $e_j$  are positive real numbers, and each  $Q_j$  belongs to a particular finite set of integer polynomials. The numbers  $e_j$  are always chosen to obtain the best auxiliary function. Such an auxiliary function was produced by Smyth [Smy84] to study the absolute trace of totally positive algebraic integers.

Certainly  $\sum_{i=1}^d f(\gamma_i) \geq dm$ , and so

$$s_1 \geq dm + \sum_{j=1}^J e_j \log \left| \prod_{i=1}^d Q_j(\gamma_i) \right|.$$

We assume now that the polynomial  $Q$  does not divide any polynomial  $Q_j$  for  $1 \leq j \leq J$ . It follows that  $\prod_{i=1}^d Q_j(\gamma_i)$  is a nonzero rational integer, since it is the resultant of  $Q$  and  $Q_j$ , and therefore

$$s_1 \geq dm. \quad (2)$$

Since the numbers  $-\gamma_i$  also lie inside the ellipse  $\mathcal{E}_a$ , we obtain the same lower bound for  $-s_1$  if  $Q(-X)$  satisfies the same condition as  $Q$ . The same method is used to obtain bounds for  $s_k$  for  $2 \leq k \leq 40$ .

If we replace the real numbers  $e_j$  by rational numbers in the auxiliary function (1), we may write

$$f(z) = \operatorname{Re}(z) - \frac{t}{h} \log |H(z)|,$$

where  $H \in \mathbb{Z}[X]$  has degree  $h$  and  $t$  is a positive real number. We wish to find a function  $f$  whose minimum  $m$  in  $\mathcal{E}_a$  is as large as possible. That is, we seek a polynomial  $H \in \mathbb{Z}[X]$  such that

$$\sup_{z \in \mathcal{E}_a} |H(z)|^{t/h} e^{-\operatorname{Re}(z)} \leq e^{-m}.$$

Now, if we suppose that  $t$  is fixed, say  $t = 1$ , it is clear that we need an effective upper bound on the quantity

$$t_{\mathbb{Z}, \varphi}(\mathcal{E}_a) := \liminf_{\substack{h \geq 1 \\ h \rightarrow \infty}} \inf_{\substack{H \in \mathbb{Z}[X] \\ \deg H = h}} \sup_{z \in \mathcal{E}_a} |H(z)|^{t/h} \varphi(z), \quad (3)$$

where we use the weight  $\varphi(z) = e^{-\operatorname{Re}(z)}$ . To find an upper bound for  $t_{\mathbb{Z}, \varphi}(\mathcal{E}_a)$ , it suffices to obtain an explicit polynomial  $H \in \mathbb{Z}[X]$ , and then use the sequence of the successive powers of  $H$ . It can be seen that in (3), if  $t$  is fixed, we have a generalization of the classical integer transfinite diameter.

Let  $K$  be a compact subset of  $\mathbb{C}$ . If  $P \in \mathbb{C}[X]$ , we put  $|P|_{\infty, K} = \sup_{z \in K} |P(z)|$ . Recall that the *integer transfinite diameter* of  $K$  is defined by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \min_{\substack{P \in \mathbb{Z}[X] \\ \deg P = n}} |P|_{\infty, K}^{1/n}.$$

It is known that if  $K = [a, b]$  is a real interval of length  $b - a \geq 4$ , then  $t_{\mathbb{Z}}(K) = (b - a)/4$ . However, if  $b - a < 4$  then  $t_{\mathbb{Z}}(K) < 1$ , but in this case the exact value of  $t_{\mathbb{Z}}(K)$  is unknown. We also recall that a polynomial  $P_n$  with integer coefficients and positive degree  $n$  is a *Chebyshev polynomial* if its supremum norm on  $K$  is smallest among all integer polynomials of the same degree, so

$$|P_n|_{\infty, K} = \min\{|P|_{\infty, K} : P \in \mathbb{Z}[X] \text{ and } \deg P = n\}.$$

(Note that  $P_n$  need not be unique.) For more details, see [BE96, FRS97, Pri05].

In order to obtain a good upper bound on  $t_{\mathbb{Z},\varphi}(\mathcal{E}_a)$  when the parameter  $t$  in (3) is fixed, in general one needs a polynomial  $H$  of rather large degree—about  $10^8$ . However, it is not possible to compute a Chebyshev polynomial of such a large degree. Instead, we employ the third author’s algorithm [Wu03] to compute Chebyshev polynomials, or at least polynomials whose supremum norm is close to minimal, of degree less than 40, then use factors of these polynomials as the  $Q_j$  in the auxiliary function (1). More generally, when  $t$  varies, we first select an initial value of  $t$  (say  $t_0 = 1$ ). We compute a polynomial  $H_1$  of small degree (usually at most 5), let  $Q_1$  be an irreducible factor of  $H_1$ , and select a positive real number  $e_1$  to optimize our auxiliary function  $f_1$ . We deduce from this the value of  $t = t_1$ , compute a new polynomial  $H_2$  and irreducible factor  $Q_2$ , then optimize  $f_2$  with respect to the two factors  $Q_1$  and  $Q_2$ . This process is continued while  $\deg(H_i) \leq 40$ . We use Smyth’s semi-infinite linear programming method [Smy84] to optimize the auxiliary function at each stage.

As in [FRSÉ06], we also obtain additional bounds on the  $s_k$  by incorporating values of the function  $g(a) := \min_{z \in \mathcal{E}_a} f(z)$ . In fact, we derive seven different sets of bounds for the  $s_k$  by considering the value of the root  $\alpha_1$  of  $P$  of largest modulus.

- In the first six cases, we suppose that  $|\alpha_1| > \sqrt{M}$ . Thus,  $\alpha_1$  is necessarily real, and we may assume that  $\alpha_1 > 0$ . The six subcases arise by selecting an integer  $j$  between 1 and 6 so that

$$t_1(j) = \frac{1}{2} + \frac{j-1}{12} \leq \frac{\log|\alpha_1|}{\log M} < \frac{1}{2} + \frac{j}{12} = t_2(j).$$

In case  $j$ , the numbers  $\gamma_2, \dots, \gamma_d$  all lie in the ellipse  $\mathcal{E}_{1/2-(j-1)/12}$ , and are therefore close to the real axis. In this case, the worst situation occurs when  $P$  has  $2d-2$  roots of modulus 1 (i.e., the corresponding numbers  $\gamma_{i,k}$  lie in  $\mathcal{E}_0$ ), one root has modulus  $M^{1/2-(j-1)/12}$ , and the remaining root has modulus  $1/M^{1/2-(j-1)/12}$ .

- In the seventh case, we assume that all the roots of  $P$  have modulus at most  $\sqrt{M}$ . In this case, the worst situation occurs when  $2d-4$  roots  $\alpha_i$  have modulus 1, two have modulus  $\sqrt{M}$ , and two have modulus  $1/\sqrt{M}$ . Here, we may assume that the first nonzero coefficient  $b_i$  of  $Q$  with  $i$  odd satisfies  $b_i > 0$ .

Note that in any case in the worst situation we have at least  $d-2$  roots of  $Q$  lying in the real interval  $(-2, 2)$ . This is a favorable situation since our auxiliary functions are most efficient on the real axis.

We describe a further improvement in the bounds for  $s_{2k}$  for  $1 \leq k \leq 20$ . Since the seventh case above produces the most important contribution to  $E_1$ , we study some relations between  $s_k$  and  $s_{2k}$  in this case. The heuristic idea is that  $s_k$  and  $s_{2k}$  cannot simultaneously lie too close to the respective bounds we computed earlier. We consider an auxiliary function  $f$  of the following type:

$$f(z) = \operatorname{Re}(z^2 - 2) + e_0 \operatorname{Re}(z) - \sum_{1 \leq j \leq J} e_j \log |Q_j(z)|, \quad (4)$$

with the same conditions as before for the numbers  $e_j$  for  $0 \leq j \leq J$  and for the polynomials  $Q_j$ . Since  $\gamma_{i,2} = \alpha_i^2 + 1/\alpha_i^2 = (\alpha_i + 1/\alpha_i)^2 - 2 = \gamma_i^2 - 2$ , we find that  $s_2 + e_0 s_1 \geq m$ . If we assume that  $s_1$  has the value  $\sigma$ , then

$$s_2 \geq m - e_0 \sigma. \quad (5)$$

We maximize the right hand side of (5), which is linear in the numbers  $e_j$ , and obtain a lower bound for  $s_2$  depending on the value of  $s_1 = \sigma$ . When  $\sigma$  increases from the lower bound of  $s_1$  computed above, say  $\sigma = -B_1$ , the bound for  $s_2$  decreases. We stop when this lower bound is less than  $-B_2$ , the lower bound for  $s_2$  determined earlier. Since the ellipse  $\mathcal{E}_a$  is symmetric, we may replace the numbers  $\gamma_i$  by the numbers  $-\gamma_i$ . If we replace  $e_0 \operatorname{Re}(z)$  by  $-e_0 \operatorname{Re}(z)$  in (4), we get the same lower bound for  $s_2$  when  $s_1$  takes the value  $-\sigma$ . We may also replace  $\operatorname{Re}(z^2 - 2)$  by  $-\operatorname{Re}(z^2 - 2)$  in (4). Then we obtain an upper bound for  $s_2$  depending on the value of  $|s_1|$ . We obtain bounds on  $s_{2k}$  depending on  $|s_k|$  in the same way for  $2 \leq k \leq 20$ , replacing  $\gamma_i$  by  $\pm\gamma_{i,k}$  and  $M$  by  $M^k$ .

### 2.3 Computing $E_1$ , $E_2$ , and $E_3$ .

Using the bounds on the numbers  $s_k$  for  $1 \leq k \leq 40$  determined in Section 2.2, we inductively obtain bounds on the coefficients  $c_k$  for  $1 \leq k \leq d$  by using Newton's formula,

$$s_k c_0 + s_{k-1} c_1 + \cdots + s_1 c_{k-1} + k c_k = 0.$$

These bounds on the  $c_k$  determine our set of reciprocal polynomials  $E_1$ .

To calculate  $E_2$ , we employ a Pascal program (in double precision) that checks additional constraints on the values of the  $s_k$  for each polynomial in  $E_1$ , and rejects any for which a required inequality fails. This program enumerates the polynomials in  $E_1$  in each of the seven cases described in Section 2.2 separately, since each case is equipped with its own bounds on the  $s_k$ . It checks that the bounds on the  $s_k$  for  $d+1 \leq k \leq 40$  are satisfied, and verifies any additional upper and lower bounds for each  $s_{2k}$ , relative to the value of  $s_k$ . In addition, we perform the following test. For polynomials generated in one of the first six cases (where  $\alpha_1 > \sqrt{M}$ ), we note that the function  $P(x)$  is convex for  $x > t_1(j)$ , with  $j$  as in Section 2.2. Thus, the line joining the points  $(t_1(j), P(t_1(j)))$  and  $(t_2(j), P(t_2(j)))$  intersects the real axis at a value  $t_3 \leq \alpha_1$ . Then all the other roots of  $P$  lie in the disk of radius  $M_1 = M/t_3$ . In all cases, we use the Schur-Cohn algorithm [Mar66] to compute the number of roots of  $P$  that lie outside a sequence of disks of decreasing radius. From this we obtain a lower bound on  $M(P)$ , and we check that this bound does not exceed  $M$ . The surviving polynomials form the set  $E_2$ , which we save to a file for the third phase of the search.

In the third phase, we use PARI [BBB<sup>+</sup>02] to implement the modified Graeffe algorithm described in [FRSÉ06], and use this to obtain a lower bound on the measure of each polynomial in the set  $E_2$ . We reject any polynomial for which we determine that  $M(P) > M$ . Then we compute the Mahler measure of each of the surviving polynomials to obtain the set  $E_3$ .

$D$	$M$	$ E_1 $	$ E_2 $	$ E_3 $	Jobs	CPU Time
42	1.2400	29 648 652 246	2 606	3	12	0.2 day
44	1.2367	91 026 218 551	3 456	1	11	0.7 day
46	1.2308	260 567 756 447	4 170	1	11	1.8 day
48	1.2323	1 284 442 516 789	6 207	1	12	8.6 days
50	1.2420	14 890 559 906 854	16 295	2	100	99.1 days
52	1.2350	37 543 959 636 961	17 155	1	128	246.5 days
54	1.2366	265 346 561 290 684	29 685	1	757	1668.5 days

Table 2: Summary of search for minimal measures to degree 54.

$D$	$M$	$ E_1 $	$ E_2 $	$ E_3 $	Jobs	CPU Time
42	1.3	1 803 973 627 644	31 318	57	85	14.8 days
44	1.3	13 329 991 800 456	53 185	77	101	125.6 days

Table 3: Summary of search for measures less than 1.3 to degree 44.

### 3 Results

We used this method to perform two searches for polynomials with small Mahler measure. First, we searched for polynomials with especially small measure by setting the value of  $M$  for each fixed degree to a real number that is slightly larger than the smallest known Mahler measure of a primitive, irreducible, non-cyclotomic, integer polynomial of that degree. Table 2 shows the value of  $M$  selected in each case in these computations, which were completed through degree 54. Second, we determined all polynomials with measure less than 1.3 and degree 42 or 44. Table 3 summarizes these calculations.

Both tables exhibit the size of each set  $E_1$ ,  $E_2$ , and  $E_3$  constructed by the method, the number of jobs used to compute  $E_2$ , and the total computation time to construct  $E_2$ , given the bounds on the  $s_k$ . These computations were distributed across multiple processors in a simple way. For smaller degrees, we allowed one processor to handle one of the cases described in Section 2.2 (although case 7 was always split over several computers). For larger degrees, each case was split among several processors by prescribing up to three initial coefficients (besides  $c_0 = 1$ ) of the polynomials examined in a particular job.

The first set of computations verified that each of the polynomials listed in Table 1 indeed has the minimal Mahler measure among the primitive, irreducible, noncyclotomic polynomials of the same degree. The second set verified that the lists of polynomials in [Mos07] with measure less than 1.3 and degree 42 or 44 are complete. Incidentally, it also established that the list of known Salem numbers less than 1.3 shown in [Mos07] is also complete to degree 44, extending the result of [FGR99], where this was verified up to degree 40.

Our computations also uncovered three new polynomials with measure less than 1.3. These polynomials are exhibited in Table 4. It is interesting that

$D$	Measure	$\nu$	Half of coefficients
46	1.2766661568	5	1 2 2 1-1-2-2-2-1 0 1 1 0-1-2-2-1 0 1 1 1 1 1 1
48	1.2960624293	6	1 2 2 1-1-2-2-2-1 0 1 1 0-1-2-2-1 0 1 1 1 1 1 1 1
52	1.2990402520	8	1 2 2 1 0-1-1-1-2-3-3-3-2-1 0 1 1 0 0 1 2 3 3 2 1 1

Table 4: New polynomials with Mahler measure less than 1.3.

$n$	$D$
5, 6, 7	181
8, 9	131
10, 11	101
12, 13	75
14, 15	55
16, 17	47
18, 19	43

Table 5: Maximum degree  $D$  checked in search of polynomials with height 1 and length  $n$ , from [Mos98].

these polynomials were not discovered in prior heuristic searches, considering that the method of searching sparse polynomials with height 1 was so successful in identifying polynomials with small Mahler measure. However, using lattice reduction we may compute sparse, height 1 multiples of each of the new polynomials, and with this we see how the prior searches missed these three examples. For example, multiplying the new degree 46 polynomial by the cyclotomic product  $\Phi_1\Phi_2^2\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{22}\Phi_{26}$ , where  $\Phi_n$  denotes the  $n$ th cyclotomic polynomial, produces a polynomial with height 1 and length 12:

$$X^{83} - X^{76} - X^{66} + X^{50} + X^{44} - X^{43} + X^{40} - X^{39} - X^{33} + X^{17} + X^7 - 1.$$

However, the search of twelve-term, height 1 polynomials in [Mos98] checked only up to degree 75. We also find height 1 multiples of this polynomial with the same measure of degree 70 and length 14 (by using  $\Phi_1\Phi_2\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{22}$ ), degree 53 and length 16 (by using  $\Phi_1\Phi_6\Phi_8$ ), and degree 49 and length 18 (by using  $\Phi_1\Phi_6$ ). Table 5 shows that none of these was covered by the earlier searches of sparse polynomials. Similar phenomena occur for the other two new polynomials. In fact, the best sparse multiple of the degree 52 polynomial we find with height 1 and the same measure has degree 73 and length 18, achieved by multiplying by the cyclotomic product  $\Phi_1\Phi_3\Phi_6\Phi_{15}\Phi_{24}$ .

## References

- [AD00] F. Amoroso and R. Dvornicich, *A lower bound for the height in abelian extensions*, J. Number Theory **80** (2000), no. 2, 260–272.

- [BBB<sup>+</sup>02] C. Batut, K. Belabas, D. Bernardi, H. Cohen, and M. Olivier, *GP-PARI version 2.1.5*, 2002.
- [BDM07] P. Borwein, E. Dobrowolski, and M. J. Mossinghoff, *Lehmer's problem for polynomials with odd coefficients*, Ann. of Math. (2) **166** (2007), no. 2, 347–366.
- [BE96] P. Borwein and T. Erdélyi, *The integer Chebyshev problem*, Math. Comp. **65** (1996), no. 214, 661–681.
- [BM05] D. W. Boyd and M. J. Mossinghoff, *Small limit points of Mahler's measure*, Experiment. Math. **14** (2005), no. 4, 403–414.
- [Bor02] P. Borwein, *Computational Excursions in Analysis and Number Theory*, CMS Books Math./Ouvrages Math SMC, vol. 10, Springer-Verlag, New York, 2002.
- [Boy80] D. W. Boyd, *Reciprocal polynomials having small measure*, Math. Comp. **35** (1980), no. 152, 1361–1377.
- [Boy89] ———, *Reciprocal polynomials having small measure, II*, Math. Comp. **53** (1989), no. 187, 355–357, S1–S5.
- [Bre51] R. Breusch, *On the distribution of the roots of a polynomial with integral coefficients*, Proc. Amer. Math. Soc. **2** (1951), no. 6, 939–941.
- [BV87] E. Bombieri and J. D. Vaaler, *Polynomials with low height and prescribed vanishing*, Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), 1987, pp. 53–73.
- [BZ01] E. Bombieri and U. Zannier, *A note on heights in certain infinite extensions of  $\mathbb{Q}$* , Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **12** (2001), 5–14.
- [DM05] A. Dubickas and M. J. Mossinghoff, *Auxiliary polynomials for some problems regarding Mahler's measure*, Acta Arith. **119** (2005), no. 1, 65–79.
- [Dob79] E. Dobrowolski, *On a question of Lehmer and the number of irreducible factors of a polynomial*, Acta Arith. **34** (1979), no. 4, 391–401.
- [EW99] G. Everest and T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*, Universitext, Springer-Verlag, London, 1999.
- [FGR99] V. Flammang, M. Grandcolas, and G. Rhin, *Small Salem numbers*, Number Theory in Progress, Vol. 1 (Zakopane-Kościelisko, 1997), 1999, pp. 165–168.
- [FRS97] V. Flammang, G. Rhin, and C. J. Smyth, *The integer transfinite diameter of intervals and totally real algebraic integers*, J. Théor. Nombres Bordeaux **9** (1997), no. 1, 137–168.
- [FRSÉ06] V. Flammang, G. Rhin, and J.-M. Sac-Épée, *Integer transfinite diameter and polynomials with small Mahler measure*, Math. Comp. **75** (2006), no. 255, 1527–1540.
- [Gar] J. Garza, *The Mahler measure of dihedral extensions*, Acta Arith. To appear.
- [GH01] E. Ghate and E. Hironaka, *The arithmetic and geometry of Salem numbers*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 3, 293–314.
- [Hir01] E. Hironaka, *The Lehmer polynomial and pretzel links*, Canad. Math. Bull. **44** (2001), no. 4, 440–451. Correction, *ibid.* **45** (2002), no. 2, 231.
- [Hir03] ———, *Lehmer's problem, McKay's correspondence, and 2, 3, 7*, Topics in Algebraic and Noncommutative Geometry (Luminy/Annapolis, MD, 2001), 2003, pp. 123–138.
- [HS93] G. Höhn and N.-P. Skoruppa, *Un résultat de Schinzel*, J. Théor. Nombres Bordeaux **5** (1993), no. 1, 185.
- [Leh33] D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. (2) **34** (1933), no. 3, 461–479.
- [Lis00] P. Lisonek, 2000. Private communication.

- [Mar66] M. Marden, *Geometry of Polynomials*, Amer. Math. Soc., Providence, R.I., 1966.
- [McM02] C. T. McMullen, *Coxeter groups, Salem numbers and the Hilbert metric*, Publ. Math. Inst. Hautes Études Sci. (2002), no. 95, 151–183.
- [Mos03] M. J. Mossinghoff, *Polynomials with restricted coefficients and prescribed noncyclotomic factors*, LMS J. Comput. Math. **6** (2003), 314–325.
- [Mos07] ———, *Lehmer’s problem*, 2007. <http://www.cecm.sfu.ca/~mjm/Lehmer>.
- [Mos98] ———, *Polynomials with small Mahler measure*, Math. Comp. **67** (1998), no. 224, 1697–1705, S11–S14.
- [MPV98] M. J. Mossinghoff, C. G. Pinner, and J. D. Vaaler, *Perturbing polynomials with all their roots on the unit circle*, Math. Comp. **67** (1998), no. 224, 1707–1726.
- [Pri05] I. E. Pritsker, *Small polynomials with integer coefficients*, J. Anal. Math. **96** (2005), 151–190.
- [RSÉ03] G. Rhin and J.-M. Sac-Épée, *New methods providing high degree polynomials with small Mahler measure*, Experiment. Math. **12** (2003), no. 4, 457–461.
- [Sch00] A. Schinzel, *Polynomials with Special Regard to Reducibility*, Encyclopedia Math. Appl., vol. 77, Cambridge Univ. Press, Cambridge, 2000.
- [Sch73] ———, *On the product of the conjugates outside the unit circle of an algebraic number*, Acta Arith. **24** (1973), 385–399. Addendum, *ibid.* **26** (1975), 329–331.
- [Sch95] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Progr. Math., vol. 128, Birkhäuser, Basel, 1995.
- [Smy07] C. J. Smyth, *The Mahler measure of algebraic numbers: a survey* (2007). arXiv:math.NT/0701397v1, 30 pp.
- [Smy71] ———, *On the product of the conjugates outside the unit circle of an algebraic integer*, Bull. London Math. Soc. **3** (1971), 169–175.
- [Smy84] ———, *The mean values of totally real algebraic integers*, Math. Comp. **42** (1984), no. 166, 663–681.
- [SW04] D. S. Silver and S. G. Williams, *Mahler measure of Alexander polynomials*, J. London Math. Soc. (2) **69** (2004), no. 3, 767–782.
- [Vou96] P. Voutier, *An effective lower bound for the height of algebraic numbers*, Acta Arith. **74** (1996), no. 1, 81–95.
- [Wal00] M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Grundlehren Math. Wiss., vol. 326, Springer-Verlag, Berlin, 2000.
- [Wu03] Q. Wu, *On the linear independence measure of logarithms of rational numbers*, Math. Comp. **72** (2003), no. 242, 901–911.

Michael J. Mossinghoff, Department of Mathematics, Davidson College, Davidson, North Carolina 28035-6996 (mimossinghoff@davidson.edu)

Georges Rhin, UMR CNRS 7122, Département de Mathématiques, UFR MIM, Université de Metz, Ile du Saulcy, 57045 Metz Cedex 01, France (rhin@univ-metz.fr)

Qiang Wu, Department of Mathematics, Southwest University of China, 2 Tiansheng Road Beibei, 400715 Chongqing, China (qiangwu@swu.edu.cn)